2017 **University of Utah**

Mail: kenkel@math.utah.edu

Local Cohomology of Determinantal Thickenings

Jenny Kenkel

Set Up:

- $\mathbb{F}$ a field
- $R = \mathbb{F}[x_1, x_2, \ldots, x_n]$ a polynomial ring
- I is a homogenous ideal
- j fixed degree

The map

$$R/I^{t+1} \to R/I^t$$

induces a map

$$H^k_{\mathfrak{m}}(R/I^{t+1})_j \to H^k_{\mathfrak{m}}(R/I^t)_j$$

Theorem (BBLSZ 2016)

- R a standard graded polynomial ring
- $char(\mathbb{F}) = 0$
- for $\mathfrak{p} \neq \mathfrak{m}$, $(R/I)_{\mathfrak{p}}$ a complete intersection

If dim Sing$(R/I) = i$, then for all $k < dim(R/I) - i$, the maps

$$H^k_{\mathfrak{m}}(R/I^{t+1})_j \to H^k_{\mathfrak{m}}(R/I^t)_j$$

are isomorphisms for $t \gg 0$.

Determinantal Example

Let R be the polynomial ring in a 2×3 matrix of indeterminates over a field $\mathbb{F}$ of characteristic 0, and consider the ideal I generated by the size two minors of the matrix

$$R = \mathbb{C}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$$

$$I = (\Delta_1 = vz - wy, \Delta_2 = wx - uz, \Delta_3 = uy - vx)$$

- $\dim(R/I) = 4$
- dim Sing$(R/I) = 0$ so the theorem applies when $k < 4$
- Determinantal rings are Cohen-Macaulay by Hochster and Eagon
- Thus R/I is Cohen Macaulay, so $H^3_{\mathfrak{m}}(R/I) = 0$
- For $t \geq 2$, R/I^t is not Cohen-Macaulay by DeConcini, Eisenbud and Procesi
- As it turns out, $H^3(R/I^t)_0$ is a rank one vector space for all $t \geq 2$

Consider

$$\begin{aligned}
0 &= \ln(1) \\
&= \ln\left(\frac{wy}{vz}\frac{uz}{wx}\frac{vx}{uy}\right) \\
&= \ln\left(\left(1 - \left(1 - \frac{wy}{vz}\right)\right)\left(1 - \left(1 - \frac{uz}{wx}\right)\right)\left(1 - \left(1 - \frac{vx}{uy}\right)\right)\right) \\
&= \ln\left(\left(1 - \left(\frac{\Delta_1}{vz}\right)\right)\left(1 - \left(\frac{\Delta_2}{wx}\right)\right)\left(1 - \left(\frac{\Delta_3}{uy}\right)\right)\right) \\
&= \ln\left(\left(1 - \left(\frac{\Delta_1}{vz}\right)\right) + \ln\left(1 - \left(\frac{\Delta_2}{wx}\right)\right) + \ln\left(1 - \left(\frac{\Delta_3}{uy}\right)\right)\right) \\
&= \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_1}{vz}\right)^m + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_2}{wx}\right)^m + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_1}{uy}\right)^m,
\end{aligned}$$

since

$$\sum_{m=1}^{\infty} \frac{1}{m} a^m = \ln(1-a)$$

This equation gives the generator for $H^3_{\mathfrak{m}}(R/I^t)_0$.

We are guaranteed an isomorphism from the BBLSZ theorem; in this example the isomorphism

$$H^3_{\mathfrak{m}}(R/I^{t+1})_0 \to H^3_{\mathfrak{m}}(R/I^t)_0$$

takes the form

$$\sum_{m=1}^{t+1} \frac{1}{m}\left(\frac{\Delta_1}{vz}\right)^m + \sum_{m=1}^{t+1} \frac{1}{m}\left(\frac{\Delta_2}{wx}\right)^m + \sum_{m=1}^{t+1} \frac{1}{m}\left(\frac{\Delta_1}{uy}\right)^m$$
$$\mapsto \sum_{m=1}^{t} \frac{1}{m}\left(\frac{\Delta_1}{vz}\right)^m + \sum_{m=1}^{t} \frac{1}{m}\left(\frac{\Delta_2}{wx}\right)^m + \sum_{m=1}^{t} \frac{1}{m}\left(\frac{\Delta_1}{uy}\right)^m$$

i.e., the power series is truncated one step earlier.

Characteristic p

Let R be as before but over a field of characteristic $p > 0$. Let q be the largest power of p such that $q < t$, and let q_2 be the smallest power of p such that $q + q_2 > t$.

Then rank$H^3_{\mathfrak{m}}(R/I^t)_0 = 2\left\lfloor \frac{q}{q_2} \right\rfloor - 1$

Table: rank $H^3_{\mathfrak{m}}(R/I^t)_0$

t	p=2	p=3	p=5
2	1	1	1
3	3	1	1
4	1	5	1
5	7	1	1
6	3	1	9
7	1	1	1
8	1	1	1
9	15	1	1
10	7	17	1
11	3	5	1
12	3	5	1
13	1	1	1

References

- B. Bhatt, M. Blickle, G.Lyubeznik, A.K. Singh, and W. Zhang, *Stabilization of the Cohomology of Thickenings*, arXiv:1605.09492.
- C. DeConcini, D. Eisenbud, and C. Procesi, *Young Diagrams and Determinantal Varieties*, Invent. Math. **56** (1980), 129–165.
- M. Hochster and J. Eagon, *Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci*, Amer. J. Math. **93** (1971), 1020–1058.